PERGAMON

International Journal of Solids and Structures 40 (2003) 3707–3716

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Solving inverse viscoelasticity problems via aggregate function approach

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Received 29 November 2001; received in revised form 9 January 2003

Abstract

This paper presents a nonlinear programming model with multi-constraints of inequality to solve inverse viscoelasticity problems. By utilizing an aggregate function approach, multi-constraints are converted into a single smooth constraint. The optimization with a single constraint is realized by using a technique of multiplier penalty functions, and a standard BFGS algorithm is employed in the solution process. Results with time dependent and independent noise data are given.

Keywords: Viscoelasticity; Inverse problem; Aggregate function; Nonlinear programming; Constitutive relationship

1. Introduction

One of the important issues of inverse viscoelasticity problems is determining unknown viscoelastic parameters via some known information of quasi-static displacements. A number of relevant literatures can be found. Li (1992) employed an iterative method to estimate Young's module of a concrete dam; Yang (1996, 1998) developed an approach identifying viscoelastic parameters in homogeneous and inhomogeneous media. A 3-D inverse analysis for identifying viscoelastic constitutive parameters was given by Yang (see e.g. Yang and Zhu, 1991). The application of six kinds optimization techniques solving inverse viscoelasticity problems was discussed by Lv (1996), the merits and demerits of these techniques were evaluated in terms of the choice of initial guess, convergence rate, convergence precision, etc.

In the past work, the determination of unknown viscoelastic parameters was usually proposed and solved as an unconstrained optimization problem. However for the constitutive parameters to be identified, there physically exist a constraint of lower bound greater than '0'. Without consideration of this constraint, numerical oscillation, lower convergence rate, and even divergence may occur in the iterative process of

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0020-7683/03/$ - see front matter © 2003 Published by Elsevier Science Ltd.
doi:10.1016/S0020-7683(03)00077-5

unconstrained optimization, especially in the case without guarantee of the convexity for the proposed problems.

With the above consideration, a nonlinear programming model with multi-constraints is proposed to estimate unknown viscoelastic parameters in this paper. By exploiting a maximum entropy theory based on aggregate function method, multi-constraints can be converted into a single differentiable constraint without distinguishing active and inactive constraints in the iterative process. The optimization with a single constraint is realized using a technique of multiplier penalty functions. Satisfactory results are shown in the numerical validation, and the effects of time dependent and independent noise data on the results are given.

2. Governing equations for direct viscoelasticity problems

For direct viscoelasticity problems, the governing equations which describe equilibrium relationship, relationship of displacement and strain, and constitutive relationship, can be given by (see e.g. Christensen, 1982)

$$[H_1]\{\sigma\} + \{F\} = 0 \tag{1}$$

$$\{\varepsilon\} = [H_2]\{u\} \tag{2}$$

$$\{\varepsilon\} = [D]L(\{\sigma\} - \{\sigma\}_0) \tag{3}$$

where $\{\varepsilon\}$ and $\{\sigma\}$ represent strain and stress vectors, $\{F\}$ is a vector of body force, $\{\sigma\}_0$ denotes a vector of initial stress, $[D]$ is a constant matrix only related with Poisson ratio μ, $[H_1] = [H_2]^{\mathrm{T}}$ denotes a matrix of differential operators.

For the plane stress problem

$$[D] = \begin{bmatrix} 1 & -\mu & 0 \\ -\mu & 1 & 0 \\ 0 & 0 & 2(1+\mu) \end{bmatrix} \tag{4}$$

$$[H_2] = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \tag{5}$$

L is an integral operator (see e.g. Christensen, 1982), and is defined by

$$L = \left\{ \frac{(\ )}{E} - \int_0^t (\ ) \frac{\partial}{\partial \tau} \delta \, \mathrm{d}\tau \right\} \tag{6}$$

where E is Young's modulus, t represents time, τ is an integral variable, and δ denotes a kernel function.

The boundary conditions are

$$\{u\} = \{\hat{u}\} \quad x \in \Gamma_u \tag{7}$$

$$[n]\{\sigma\} = \{f\} \quad x \in \Gamma_\sigma \tag{8}$$

where $\{\hat{u}\}$ is a vector of prescribed displacements, $[n]$ refers to a unit vector of outside normal on the boundaries $\{f\}$ is a vector of prescribed traction on the boundaries, x represents a vector of coordinates

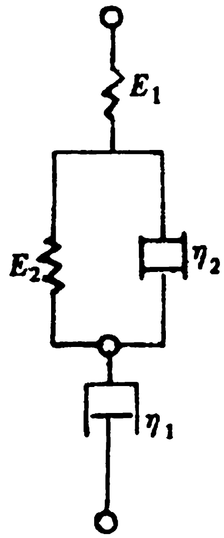

Fig. 1. A Burgers model.

$\Gamma = \Gamma_u + \Gamma_\sigma$ denotes the whole boundary of the domain, subscripts u and σ represent displacement and stress, respectively.

Under the condition that $\{F\}$, $\{f\}$, and $\{\sigma\}_0$ are time independent, and $\{\hat{u}\} = 0$, it has been proved that the solution of Eqs. (1)–(3), (7) and (8) can be given by (see e.g. Yang and Li, 2000)

$$\{u\} = Q(X,t)\{u\}^{\mathrm{e}} \tag{9}$$

where $\{u\}^{\mathrm{e}}$ is the solution of an elasticity problem, and can be determined by $\{F\}$, $\{f\}$ and $\{\sigma\}_0$ analytically or numerically.

$$Q(X,t) = 1/E - \int_0^t \frac{\partial}{\partial \tau}\delta(t,\tau,X)\,\mathrm{d}\tau \tag{10}$$

where X represents a vector of viscoelastic parameters, $\delta(t,\tau,X)$ is a kernel function, and has different forms for different viscoelastic models. In this paper, Burgers model is adopted (as shown in Fig. 1). In this case, $Q(x,t)$ becomes

$$Q(t,x) = x_1 + t\cdot x_2 + (1 - \exp(-x_4\cdot t))\cdot x_3 \tag{11}$$

where $x_1 = 1/E_1$, $x_2 = 1/\eta_1$, $x_3 = 1/E_2$, $x_4 = E_2/\eta_2$.

The reduction of model (11) leads to some simpler viscoelastic models, i.e.

(a) $Q(x,t)$ will tend to be a Maxwell model when E_2 and η_2 approach to ∞,
(b) $Q(x,t)$ will tend to be a Kelvin model when E_1 and η_1 approach to ∞,
(c) $Q(x,t)$ will tend to be a Linear model when η_1 approaches to ∞.

3. Inverse viscoelasticity problem

For the inverse problem of Eq. (9), the left hand side of Eq. (9) are all or partially known, the unknowns to be determined are $\{X\} = \{x_1, x_2, x_3, x_4\}$ on the right hand side.

$\{X\}$ can be evaluated by minimizing a function defined by

$$\Pi = \frac{1}{2}\sum(\tilde{u}_k - u_k^*)^{\mathrm{T}}(\tilde{u}_k - u_k^*) = \sum R_k^{\mathrm{T}} \cdot R_k \tag{12}$$

The constraints can be described by

$$\text{S.T.} \quad x_i > 0 \quad (i = 1, 2, \ldots, m) \tag{13}$$

where u_k^* denotes a vector of known quasi-static displacements which is usually obtained by measurement; $\tilde{u}_k$ is given by Eq. (9), and subscript k represents a time series of sample points. The constraints of Eq. (13) represent the physical requirements for the viscoelastic parameters.

The sensitivity of Π with respect to $\{X\}$ is given by

$$\frac{\partial \Pi}{\partial X} = \sum G_k^{\mathrm{T}} \cdot R_k$$

where

$$G_k = \frac{\partial \tilde{u}_k}{\partial X} = \{u^{\mathrm{e}}\} \cdot \frac{\partial Q_k(X,t)}{\partial X} \tag{14}$$

$$\frac{\partial Q_k(X,t)}{\partial x_1} = 1 \tag{15}$$

$$\frac{\partial Q_k(X,t)}{\partial x_2} = t_i \tag{16}$$

$$\frac{\partial Q_k(X,t)}{\partial x_3} = 1 - \exp(-x_4 \cdot t_i) \tag{17}$$

$$\frac{\partial Q_k(X,t)}{\partial x_4} = x_3 t_i \exp(-x_4 \cdot t_i) \tag{18}$$

4. Implementation of aggregate function method

Multi-constraints defined by Eq. (13) can be converted into a single differentiable constraint via a maximum entropy theory based on aggregate function method (see e.g. Li, 1991, 1994), the trouble caused by distinguishing active and inactive constraints in the iterative process can therefore be avoided. Furthermore, some well developed algorithms, such as quasi-exact penalty function algorithm, multiplier penalty functions algorithm etc., can be exploited (see e.g. Li, 1991, 1994).

Consider a problem defined by

$$(\mathrm{P}) \quad \begin{cases} \min & f(Y) \\ \text{s.t.} & g_i(Y) \leqslant 0, \quad i = 1, 2, \ldots, m \end{cases} \quad Y \in R^n \tag{19}$$

where Y is a vector of variables, $f(Y)$ and $g_i(Y)$ are smooth nonlinear functions of Y.

The problem (P) can be converted into an equivalent problem with a single constraint

$$(\mathrm{P1}) \quad \begin{cases} \min & f(Y) \\ \text{s.t.} & \gamma(Y) \leqslant 0 \end{cases} \tag{20}$$

where the single constraint is termed as 'maximum' constraint, having the form

$$\gamma(Y) = \max_i \{g_i(Y)\} \tag{21}$$

The problems of (P1) and (P) are definitely equivalent since they have the same feasible regions. Due to the nondifferentiability of Eq. (21), g_p, a 'surrogate constraint' or 'aggregate function', was proposed by Li (1991, 1994) to smooth constraint, it can be described by

$$g_p(Y) = (1/p)\ln\left\{\sum_{i=1}^{m}\exp[p\cdot g_i(Y)]\right\} \tag{22}$$

where p is a positive parameter.

There exists an inequality relationship (see e.g. Li, 1994)

$$\gamma(Y) \leqslant g_p(Y) \leqslant \gamma(Y) + \ln(m)/p \tag{23}$$

where m is the number of constraints.

Li (1994) proved that $g_p(Y)$ will approach $\gamma(Y)$ uniformly in R^n when p tends to infinity. Thus the problem (P1) with a nonsmooth constraint (21) can be equivalent to a problem with a single smooth constraint, i.e.

$$(\mathrm{P2}) \quad \begin{cases} \min & f(Y) \\ \text{s.t.} & g_p(Y) = (1/p)\ln\left\{\sum_{i=1}^{m}\exp[p\cdot g_i(Y)]\right\} \leqslant 0 \end{cases} \tag{24}$$

$g_p(Y)$ represents an integral effect of all constraints. The adoption of $g_p(Y)$ can make computing more efficient (see e.g. Li, 1994).

When problem (P) has at least one 'active' constraint, the single inequality constraint of (P2) can be further written as an equality constraint (see e.g. Cheng, 1984; Tang and Qin, 2000). By means of multiplier penalty functions (see e.g. Tang and Qin, 2000), the problem (P2) can be treated as an unconstrained optimization defined by

$$(\mathrm{P3}) \quad \min \quad \Phi_p(Y,\alpha) = f(Y) + \alpha\cdot g_p(Y) + c\cdot g_p^2(Y)/2 \tag{25}$$

where c is a penalty factor, α is a Lagrange multiplier associated with the single constraint (22), α is equal to the sum of all Lagrange multipliers in the problem (P) (see e.g. Li, 1994).

In the iterative process, α will be updated by

$$\alpha^{k+1} = \alpha^k + c\cdot g_p(Y^k) \tag{26}$$

In order to solve Eq. (25), a standard BFGS algorithm (see e.g. Cheng, 1984; Tang and Qin, 2000) for unconstrained optimization is employed.

The major steps of solving Eq. (25) via the BFGS algorithm include

I. Set n (the number of unknown variables), ε (the convergence precision), Y^0 (the initial guess), p and c.
   Set $\alpha_0 = 0$, $B_0 = I$ (unit matrix), and $K = 0$.
   Calculate F_0, the gradient of the objective function (25) at the point Y^0.
II. Set $S^K = -B_K^{-1}F_K$,
   Determine α^K by minimizing $f(Y^k + \alpha^k S^k)$ along the direction of S^K,
   Set $Y^{K+1} = Y^K + \alpha^K S^K$
   Calculate F_{K+1}
III. Check the criterion if $\|Y^{k+1} - Y^k\| \leqslant \varepsilon$, Then
   $Y^{*B} = Y^{K+1}, \quad f^{*B} = f(Y^{K+1})$
   stop iteration, and go to VI
   Else Go to IV
IV. Calculate

$$B_{K+1} = B_k + \left[1 + \frac{\gamma_K^T B_K \gamma_K}{\gamma_K^T \gamma_K}\right] \frac{\delta_K \delta_K^T}{\delta_K^T \gamma_K} - \frac{\delta_K \gamma_K B_K + B_K \gamma_K \delta_K^T}{\delta_K^T \gamma_K}$$

$$\delta_K = Y^{K+1} - Y^K, \quad \gamma_K = F^{K+1} - F^K$$

V. Update α by using Eq. (26)
Set $K = K + 1$
Go to II

VI. Stop

where F_i represents the gradient of the objective function (25) at the point Y^i.

5. Numerical examples and remarks

By considering Eqs. (12) and (13) as Problem (P), and using the techniques proposed in the Section 4, a number of numerical tests are carried out. The results of identification are exhibited in Table 1. Table 2 shows the effect of initial guess of iteration on the results

Two kinds of noise data are taken into account (see e.g. Wang et al., 2000), i.e.

$$\{u^*\} = Q(X,t)(1 + \sigma \cdot \xi) \cdot \{u^e\} \quad \text{(time independent noise data)} \tag{27}$$

and

$$\{u^*\} = Q(X,t) \cdot (1 + \sigma \cdot \xi \cdot \sin(t)) \cdot \{u^e\} \quad \text{(time dependent noise data)} \tag{28}$$

where $\{u^*\}$ represents the known information of quasi-static displacements with noise data, ξ is a random variable, and follows a normal distribution with zero mean and unit standard deviation, σ denotes a deviation.

For each fixed value of σ, 50 groups of results are obtained with 50 ξ produced randomly.The confidence interval is evaluated by (see e. g. Wang et al., 2000)

$$\bar{x} \pm \frac{t_{(\beta/2,N-1)} * S}{\sqrt{N}} \tag{29}$$

where $\bar{x}$ represents the mean of identified parameters, S is the standard deviation of identified parameters, t denotes a t distribution with the degree of freedom $(N - 1)$, N is the capability of samples, and the confidence level is $1 - \beta$.

The results with a confidence interval of 95% for both time independent and dependent noise data are given in Tables 3 and 4 where all the computing parameters are as same as those in Table 1.

In the above numerical examples,

$$\{u^e\}^T = (1.0000, 1.0005, 1.0006),$$

the units adopted are time: s (second), E: N/cm^2, η: s N/cm^2.

On the basis of the above numerical tests, some remarks can be given as follows

(i) The proposed approach is capable of identifying viscoelasticity constitutive parameters/models within few steps of iteration.
(ii) The choice of initial guess shows slight effect on the final results, however steps of iteration will be affected.
(iii) The results of identification are basically not affected by choosing different number of sample points.

Table 1
Identification of viscoelastic parameters/models

Initial guesses				Final values				Actual values				Size of time interval	Number of sample points	Number of iterations	Models
E_1	η_1	E_2	η_2	E_1	η_1	E_2	η_2	E_1	η_1	E_2	η_2				
1.0E8	1.0E9	1.0E8	1.0E10	1.00E5	1.00E8	6.73E18	6.73E21	1.00E5	1.00E8	∞	∞	5. 0	878	2	Maxwel
1.0E8	1.0E9	1.0E8	1.0E10	0.24E9	0.21E13	5.00E4	1.00E7	∞	∞	5.00E4	1.00E7	5. 0	578	2	Kelvin
1.0E8	1.0E9	1.0E8	1.0E10	1.00E5	7.99E11	4.99E4	0.99E7	1.00E5	∞	5.00E4	1.00E7	5. 0	578	3	Linear
1.0E8	1.0E9	1.0E8	1.0E10	1.00E5	0.99E8	4.99E4	0.997E7	1.00E5	1.00E8	5.00E4	1.00E7	5. 0	578	3	Burgers

Table 2
The effect of initial guess on the results

Initial guesses				Final values				Actual values				Size of time interval	Number of sample points	Number of iterations
E_1	η_1	E_2	η_2	E_1	η_1	E_2	η_2	E_1	η_1	E_2	η_2			
1.0E11	1.0E15	1.0E12	1.0E16	0.998E5	1.000E8	5.000E4	1.000E7	1.0E5	1.0E8	5.0E4	1.0E7	5.0	578	15
1.0E6	1.0E10	1.0E10	1.0E12	1.011E5	0.999E8	4.980E4	0.982E7	1.0E5	1.0E8	5.0E4	1.0E7	5.0	778	15
1.0E4	1.0E5	1.0E4	1.0E6	1.004E5	0.999E8	4.993E4	0.993E7	1.0E5	1.0E8	5.0E4	1.0E7	5.0	678	3

Table 3
The effect of time independent noise data on the results

Models		Expected values				Confidence intervals			
		E_1	η_1	E_2	η_2	E_1	η_1	E_2	η_2
Maxwel	$\sigma = 0.01$	0.995301E5	0.995301E8	0.530907E19	0.530907E21	0.993359E5–0.997243E5	0.993359E8–0.997243E8	0.470065E19–0.591748E19	0.470065E21–0.591748E21
	$\sigma = 0.03$	0.884810E5	0.986771E8	0.412023E19	0.412023E21	0.854451E5–0.915169E5	0.981132E8–0.992411E8	0.333391E19–0.490655E19	0.333391E21–0.490655E21
Linear	$\sigma = 0.01$	0.100043E6	0.451482E12	0.496967E5	0.987215E7	0.998704E5–0.100216E6	0.362738E12–0.540226E12	0.495961E5–0.497973E5	0.985218E7–0.989213E7
	$\sigma = 0.03$	0.996059E5	0.884086E12	0.492168E5	0.972402E7	0.992059E5–0.100006E6	0.821734E12–0.946438E12	0.489107E5–0.495229E5	0.963775E7–0.981029E7
Burgers	$\sigma = 0.01$	0.100680E6	0.981596E8	0.500010E5	0.978376E7	0.100578E6–0.100782E6	0.976520E8–0.986673E8	0.499614E5–0.500405E5	0.972530E7–0.984222E7
	$\sigma = 0.03$	0.100417E6	0.963926E8	0.496078E5	0.957988E7	0.100235E6–0.100599E6	0.953527E8–0.974325E8	0.493769E5–0.498386E5	0.946076E7–0.969900E7

Table 4
The effect of time dependant noise data on the results

Models		Expected values				Confidence intervals			
		E_1	η_1	E_2	η_2	E_1	η_1	E_2	η_2
Linear	$\sigma = 0.01$	0.100748E6	0.168685E12	0.498987E5	0.988381E7	0.100641E6–0.100855E6	0.158346E12–0.179024E12	0.498839E5–0.499135E5	0.986792E7–0.989970E7
	$\sigma = 0.03$	0.100742E6	0.155425E14	0.498970E5	0.988898E7	0.100643E6–0.100841E6	0.116479E14–0.194371E14	0.498857E5–0.499084E5	0.987077E7–0.990719E7
Burgers	$\sigma = 0.01$	0.100951E6	0.986326E8	0.501735E5	0.983836E7	0.100855E6–0.101047E6	0.984981E8–0.987670E8	0.501565E5–0.501906E5	0.982185E7–0.985487E7
	$\sigma = 0.03$	0.101409E6	0.979825E8	0.502520E5	0.976649E7	0.101073E6–0.101745E6	0.975524E8–0.984126E8	0.501963E5–0.503077E5	0.970790E7–0.982509E7

(iv) The results of identification are affected by the noise data, and seem more sensitive to time dependant noise data.
(v) The choice of p is a key factor affecting computing efficiency and final results. A larger p may lead to a smaller convergence domain. A proper way is to modify p in the iteration process.
(vi) The choice of c seems no effect on the convergence, and may affect the convergence rate slightly.
(vii) The initial choice of α has great effect on the computing efficiency, by initializing $\alpha = 0$ satisfactory results have been obtained for the most numerical tests.

6. Conclusion

With the consideration of the constraints of lower bound for viscoelastic constitutive parameters, a nonlinear programming model with multi-constraints is proposed in this paper to solve inverse viscoelasticity problems. The model presented is not only more rigorous physically, but also may avoid the numerical oscillation caused by the occurrence of negative values in the iterative process.

By virtue of the aggregate function method, the above nonlinear programming problem with multi-constraints is converted into an optimization with a single smooth constraint, and is solved with satisfactory results.

Fairly good performance can be observed in the numerical tests with time independent and dependent noise data.

Due to its smooth and differentiable properties, the aggregate function method can be expected to be utilized to solve larger scale inverse problems with a larger number of constraints (see e.g. Li, 1991).

Acknowledgements

The research leading to this paper is funded by the oversea returnee's initiating fund [1999-363], the key project fund [99149], and backbone faculty fund [2000-65], all of these funds came from National Education Department of P.R. China. The research is also funded by opening fund of State Key Lab of Structural Analysis of Industrial Equipment [GZ9814], Key project fund of NSF(10032030), NSF(10172024), and NKBRSF [G1999032805].

Thanks go to Prof. D.A. Tortorelli, and Department of Mechanical and Industrial Engineering, University of Illinois at Urbana and Champaign, USA for their providing all convenience to complete the framework of this article.

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